# Problem Statement[1]

Let a motion of the pursuer evolve in three-dimensional Euclidean space $R^3$ and its dynamics be subject to the equation

$$\ddot{x} + \alpha\dot{x} = \rho u, \ \|u\| \leq 1 \tag{1}$$

where $x = (x_1, x_2, x_3)$ are geometric coordinates of the object. Here $x_1$, $x_2$ denote coordinates in the horizontal plane and $x_3$ a height. Vectors $\dot{x} = \frac{dx}{dt}$ and $\ddot{x} = \frac{d^2x}{dt^2}$ are velocity and acceleration, respectively; $\alpha$ - friction coefficient; $\rho > 0$ - resource coefficient; $u$ - control, which is chosen in a unit ball centered at the origin of $R^3$; $\|x\| = \sqrt{(x,x)}$, where by $(\cdot,\cdot)$ is denoted a scalar product of vectors.

It is assumed that control $u(t)$, $t \geq 0$, is Lebesgue measurable function of time. For simplicity'sake and in view of possible practical applications, it may be assumed that function $u(t)$ is piecewise-continuous or even piecewise-constant.

The evader moves in the horizontal plane and his motion is described by the equation

$$\ddot{y} + \beta\dot{y} = \sigma\vartheta, \ \|v\| \leq 1 \tag{2}$$

where $y = (y_1, y_2)$ are coordinates of the object. Vectors $\dot{y}$ and $\ddot{y}$ denote velocity and acceleration of the evader at point $y$, $\beta > 0$ - coefficient of friction, $\sigma > 0$ - coefficient of resources, and $\vartheta$ - control of the evader, taking its values in a flat circle centered at the origin. In the sequel we shall sometimes write $\tilde{y} = (y_1, y_2, 0)$ or even $\tilde{y} = (y, 0)$ in order to treat $y$ as vector in $R^3$.

The game (1), (2) will be analyzed from the pursuer's point of view. His goal is to achieve "soft meeting" with the evader at a finite instant of time:

$$\|x - \tilde{y}\| \leq \varepsilon_1, \ \left\|\dot{x} - \dot{\tilde{y}}\right\| \leq \varepsilon_2 \tag{3}$$

where $\varepsilon_1$, $\varepsilon_2$ are positive numbers, specifying the required proximity of the players. The hyperplane $\{y_3 = 0\}$ stands for state constraint of the pursuer. The pursuer is allowed to move in this hyperplane, not intersecting it.

[1] Arkadii A. Chikrii, "Soft Landing of Moving Objects", Cybernetics Institute, Ukraine, Kyiv-1998

Without loss of generality, the initial state of the pursuer is assumed to lie in the upper halfspace, that is $x_3^0 = x_3(0) > 0$.

To simplify the treatment, we set $\varepsilon_1 = \varepsilon_2 = 0$, that is we shall study the precise "soft landing". Note, that it is easy to pass from this problem to the problem (1)-(3) and the solution of problem (1)-(3) immediately follows from the solution of problem on precise "soft landing".

For the sake of convenience, let us reduce the second order system (1), (2) to a system of first order but yet of larger dimension with the help of introduction of new variables

$$z_1 = x,\ z_2 = \dot{x},\ z_3 = \tilde{y},\ z_4 = \dot{\tilde{y}}$$

Differentiating the above equalities in time and taking into account the equations (1), (2) we obtain an equivalent system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\alpha z_2 + \rho u \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -\beta z_4 + \sigma\vartheta \end{aligned} \tag{4}$$

In the strict sense (4) is a system of $12^{th}$ order, but, in fact, only of $10^{th}$, since although vectors $z_1$, $z_2$, $z_3$, $z_4$ are three-dimensional, $z_3$ and $z_4$ have zero third components.

Thus, the pursuer strives to bring a trajectory of system (4) to a linear subspace

$$M_0 :\ z_1 = z_3,\ z_2 = z_4 \tag{5}$$

or to a certain its neighbourhood for any admissible counteraction of the evader. In order to formulate the problem (4), (5) in a more general form and to develop a general approach for solution of the linear game problem we shall present the motion of a conflict-controlled process in the form

$$\dot{z} = Az + u - \vartheta,\ u \in U,\ \vartheta \in V,\ z \in R^n \tag{6}$$

where $A$ is a square matrix of order $n$, $U$, $V$ are nonempty compacts, and the terminal set is a cylindrical set

$$M^* = M_0 + M \tag{7}$$

Here $M_0$ is a linear subspace of $R^n$ and $M$ is a compact from the orthogonal complement $L$ to $M_0$ in $R^n$. By $\pi$ is usually denoted the operator of orthogonal projection from $R^n$ to $L$.

One can easily see that the problem of "soft landing", formulated in the form (1)-(3) or (4), (5), is a specific case of the differential game (6), (7). Pontryagin's condition for the problem (6), (7) means the nonempty of set-valued mapping

$$W(t) = \pi e^{tA} U \overset{*}{-} \pi e^{tA} V \neq \emptyset \; \forall t \geq 0 \tag{8}$$

The availability of information on a current state of the game to the pursuer will be specified separately for each particular method, presented in the paper. Denote states of the players (1), (2) by

$$\bar{x} = (x, \dot{x}) = (z_1, z_2), \quad \bar{y} = (\tilde{y}, \dot{\tilde{y}}) = (z_3, z_4)$$