Linear Differential Games of Pursuit with Integral Block of Control in its Dynamics

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January 11, 2001

1 Abstract

The game problem of bringing a trajectory of dynamic system to the terminal set, which has cylindrical form, is treated. Here the case is analyzed, when controls enter the system equation in integral form. Sufficient conditions for the game termination in some guaranteed time are derived on the basis of the Method of Resolving functions. The result is supported by a model example (see section 5) and is compared with the game "simple motions".

2 Problem Statement

We consider the dynamical process of the form

$$\frac{dz}{dt} = A(t)\, z(t) + \int_{t_0}^{t} B(t,s)\, \varphi(u(s), v(s))\, ds,\ z(t_0) = z_0, \tag{1}$$

evolving in condition of conflict. Here the phase vector z takes its values in the finite-dimensional Euclidian space R^n; $A(t)$ is n square matrix, continuously depending on $t,\ t \geq t_0$, and $B(t,s),\ t_0 \leq s \leq t < \infty$ is a matrix function, continuous in all its variables. The block of control is defined by function $\varphi(u,v)$, continuous on the direct product of compacts U and V. The pair (t,z) will be referred to as a current state of the process, and (t_0, z_0) – as its initial state. As admissible controls the players employ Lebesgue-measurable functions $u(s)$ and $v(s)$ with values in the sets U and V respectively. By virtue of the above assumptions function $\varphi(u,v)$ satisfies the condition on superpositional measurability and $\psi(s) = \varphi(u(s), \upsilon(s))$ is a bounded measurable function.

In addition, the terminal set, having cylindrical form, is given:

$$M^* = M_0 + M, \tag{2}$$

Here M_0 is a linear subspace in R^n and M is a convex compact from L, which is the orthogonal complement to M_0 in R^n.

One can easy see that in the case $B(t,s) = \delta(t-s)E$, where $\delta(t-s)$ is δ-function and E is a unit matrix, the conflict-controlled process (1), (2) reduces to ordinary differentional game [1, 2, 3].

We study the problem of bringing a trajectory of system (1) to the terminal set (2) in a some guaranteed time. In so doing, the first player (u) employs quasi-strategies, that is, at each current instant of time he constructs his control in the form

$$u(t) = u(t_0, z_0, \upsilon_t(\cdot)),$$

where $\upsilon_t(\cdot) = \{\upsilon(s) : s \in [t_0, t]\}$, for any control υ of the second player [1, 2, 3].

3 Lemma

For any chosen admissible controls of the players solution of system (1) may be presented in the form

$$z(t) = \Phi(t, t_0) z_0 + \int_{t_0}^{t} C(t,s) \varphi(u(s), \upsilon(s)) ds, \tag{3}$$

where $C(t,s) = \int_s^t \Phi(t,\tau) B(\tau, s) d\tau$, and $\Phi(t, t_0)$ is the fundamental matrix of homogeneous system (1).

Proof. From formula Cauchy [1] as applied to system (1) it follows

$$z(t) = \Phi(t, t_0) z_0 + \int_{t_0}^{t} \Phi(t,\tau) \int_{t_0}^{\tau} B(\tau, s) \varphi(u(s), \upsilon(s)) ds\, d\tau.$$

Then, using Fubini theorem [4] we have

$$z(t) = \Phi(t, t_0) z_0 + \int_{t_0}^{t} \left(\int_{s}^{t} \Phi(t, \tau) B(\tau, s) \, d\tau \right) \varphi(u(s), \upsilon(s)) \, ds,$$

whence follows formula (3).

Denote by π the orthoprojector, acting from R^n onto L. Let us study the set-valued mappings

$$W(t, s, \upsilon) = \pi C(t, s) \varphi(U, \upsilon),$$

$$W(t, s) = \bigcap_{\upsilon \in V} W(t, s, \upsilon),$$

where $\varphi(U, \upsilon) = \{\varphi(u, \upsilon) : u \in U\},\ t \geq s \geq t_0,\ \upsilon \in V$.

In the sequel, Pontryagin's condition is assumed to hold

$$W(t, s) \neq \emptyset \quad \forall (t, s) \in \Delta = \{(t, s) : t_0 \leq s \leq t < \infty\}. \tag{4}$$

By virtue of the assumptions on parameters of process (1), (2) and condition (4) the mapping $W(t, s)$ has at least a single measurable selection $\gamma(t, s)$ [5]. Fix it and set

$$\xi(t, z, \gamma(t, \cdot)) = \pi \Phi(t, t_0) z + \int_{t_0}^{t} \gamma(t, s) \, ds.$$

Let us introduce the resolving function by the formula

$$\begin{aligned} &\alpha(t, s, z, \upsilon, \gamma(t, \cdot)) = \sup \{\alpha \geq 0 : [W(t, s, \upsilon) - \gamma(t, s)] \cap \\ &\cap \alpha [M - \xi(t, z, \gamma(t, \cdot))] \neq \emptyset\}, \end{aligned} \tag{5}$$

Define the set-valued mapping

$$T(t_0, z, \gamma(\cdot, \cdot)) = \left\{ t \geq t_0 : \int_{t_0}^{t} \inf_{\upsilon \in V} \alpha(t, s, z, \upsilon, \gamma(t, \cdot)) \, ds \geq 1 \right\}. \tag{6}$$

The properties of similar functions are thoroughly studied in [3]. We only note that $T(t_0, z, \gamma(\cdot, \cdot)) = \emptyset$, if inequality in (6) fails for finite $t \geq t_0$.

4 Theorem

Let for the game problem (1), (2) Pontryagin's condition hold.

Then, if a measurable selection $\gamma(t,s) \in W(t,s)\,,\ (t,s) \in \Delta$ exists such that $T \in T(t_0, z_0, \gamma(\cdot,\cdot)) \neq \emptyset$ then a trajectory of the process (1) may be brought in a finite time from the initial state (t_0, z_0) to set (2). In so doing the first player employs quasi-strategies.

The proof is conducted by the scheme, presented in [3].

By way of illustration below is given a simple example.

5 Model Example

Let $A(t) \equiv 0,\ B(t,s) \equiv E,\ \varphi(u,v) = u - v,\ M^* = \{0\},\ U = aS,\ a > 1,$ $V = S$, where S is a square ball in R^n, centered at the origin. Set $t_0 = 0$.

Thus, a trajectory of the process

$$\frac{dz}{dt} = \int\limits_0^t (u(s) - v(s))\,ds,\ u \in aS,\ v \in S,$$

should be brought in a finite time into the origin.

In our case $M_0 = \{0\}$ and $M = \{0\}$ therefore $L = R^n$ and the orthoprojector π is an operator of identical transform and defined by the unit matrix. In the turn, since $A(t) \equiv 0$ then $\Phi(t,0) = E$.

Then $C(t,s) = (t-s)E$ and the following presentation for the set-valued mapping is true

$$W(t,s,v) = (t-s)(aS - v)\,,$$

$$W(t,s) = (t-s)(a-1)S.$$

Therefore Pontryagin's condition holds if $a \geq 1$ and $(t,s) \in \Delta$. Since $0 \in W(t,s)$, $(t,s) \in \Delta$ then we can pick $\gamma(t,s) \equiv 0$. From formula (5) we deduce that function $\alpha(t,s,z,v,0)$ appears as the greatest root of the quadratio equation

$$\|(t-s)v - \alpha z\| = (t-s)a$$

and has the form

$$\alpha\left(t,s,z,\upsilon,0\right)=\left(t-s\right)\alpha\left(z,\upsilon\right),$$

where

$$\alpha(z,\upsilon)=\frac{(z,\upsilon)+\sqrt{(z,\upsilon)^2+\|z\|(a^2-\|\upsilon\|^2)}}{\|z\|^2}.$$

Minimum of function $\alpha(z,\upsilon)$ in υ is furnished by the element $\upsilon=-\frac{z}{\|z\|}$. Then

$$\min_{\upsilon\leq 1}\alpha\left(t,s,z,\upsilon,0\right)=\left(t-s\right)\frac{a-1}{\|z\|}$$

and therefore

$$t_*=\min\left\{t\geq 0:t\in T\left(0,z,0\right)\right\}$$

is a root of the equation

$$\int_0^t\left(t-s\right)\frac{a-1}{\|z\|}ds=1.$$

Thus,

$$t_*=\left(\frac{2\|z\|}{a-1}\right)^{\frac{1}{2}}.$$

Note [3] that in the case of simple motions

$$\frac{dz}{dt}=u-\upsilon,\ u\in aS,\ \upsilon\in S,$$

the time of hitting the origin is given by the expression

$$t^*=\frac{\|z\|}{a-1}.$$

One can easy see that times t_* and t^* differ essentially.

6 Copyright Notice

7 References and Notes

References

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